

Motional Entanglement with Trapped Ions and a Nanomechanical Resonator

F. Nicacio and K. Furuya

Instituto de Física “Gleb Wataghin”, Universidade Estadual de Campinas, 13083-970, Campinas, São Paulo, Brazil

F. L. Semião

Centro de Ciências Naturais e Humanas, Universidade Federal do ABC, 09210-170, Santo André, São Paulo, Brazil

We study the entangling power of a nanoelectromechanical system (NEMS) simultaneously interacting with two separately trapped ions. To highlight this entangling capability, we consider a special regime where the ion-ion coupling does not generate entanglement in the system, and any resulting entanglement will be the result of the NEMS acting as an entangling device. We study the dynamical behavior of the bipartite NEMS-induced ion-ion entanglement as well as the tripartite entanglement of the whole system (ions+NEMS). We found some quite remarkable phenomena in this hybrid system. For instance, the two trapped ions initially uncorrelated and prepared in coherent states can become entangled by interacting with a nanoelectromechanical resonator (also prepared in a coherent state) as soon as the ion-NEMS coupling achieve a certain value, and this can be controlled by external voltage gate on the NEMS device. By considering the NEMS in an initial thermal state, we numerically show that there is not a temperature threshold above which bipartite ion-ion entanglement ceases. A distinct effect occurs when the NEMS interacts with a thermal reservoir, above a certain value of temperature, the NEMS induction of ion-ion entanglement ceases. We also show that tripartite entanglement presents a more pronounced robustness against the destructive effects of dissipation when compared to the bipartite content.

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I. INTRODUCTION

During the last years, theoretical and instrumental developments in physics have led to an unprecedented level of control over atomic and nanoscale systems. Striking examples include the manipulation of single quantum systems such as laser cooled trapped ions [1], and the advances on fabrication, characterization and application of nano- mesoscale devices such as electromechanical resonators [2]. Isolated single quantum systems are the natural choice when quantum logic is involved [3] and the nanoscale devices are recognizedly known as very sensitive weak signal detectors [4–6]. The emerging field of quantum technology takes advantage of the strong and different features of these two kind of systems by combining them together in a single setup [7–9].

The goal of quantum technology is to explore legitimate quantum resources such as entanglement to perform tasks with no classical analogue and to enforce it on nano- mesoscopic systems which can be integrated, for instance, on a chip for massive use. In order to achieve that, one very attractive route consists of cooling the NEMS down to its ground state [10] and then enforce on it a legitimate quantum behavior by coupling it to well controlled small quantum systems. For example, the coupling of qubits to these mechanical nanoresonators allows one to generate superpositions of macroscopically distinguishable states in the motion [11]. These states may be useful to study the so called quantum-to-classical transition and the phenomenon of decoherence happening in nano- and mesosystems. This is a natural follow up to the studies started in the scope of cavity quantum electrodynamics [12]. On the other hand, potential applications of NEMSs include several uses as a sensitive detector. Detection of single spins [4] and spin-spin interactions [5] as well as single molecule mass spectrometry [13] offer new possibilities in chemical imaging and characterization. For instance,

nitrogen-vacancy centers in a diamond tip coupled to NEMS resonators operating at room temperature [14] offers a new route to detect a chemical element that carry a nuclear magnetic moment. This device is sensitive enough to determine the element identity and arrangement in a complex molecule. In the case of mass spectroscopy, NEMS resonators are believed to enable in the future a detection device operating with resolution better than 1 Dalton (the hydrogen atom mass) [15]. Nowadays, the state-of-the-art is that single gold atoms can be measured [15].

Trapped ions are well studied quantum systems where experimental control of coherent quantum dynamics, state preparation, and measurement are achieved with great precision and good fidelity [1]. As a matter of fact, trapped ions have been demonstrated to achieve an error probability per randomized single-qubit gate below the threshold estimate commonly considered sufficient for fault-tolerant quantum computing [16]. This achievement of low single-qubit-gate errors is an essential step toward scalable quantum computers and simulators. It is then clear that a hybrid system composed of trapped ions and NEMS resonators represent an interesting platform to study the interplay of quantum mechanical effects in atomic and nanoscopic systems. Following this idea, one proposal [7] suggest the use of a trapped ion as a probe or device controller acting on the NEMS. In another proposal [8], trapped ions are applied to monitor and manipulate the number state of a NEMS resonator such that it enables a statistical inference of the mean phonon number of the oscillator.

Here, we take a step forward and propose the use of a NEMS resonator to generate entanglement between two trapped ions. This is a conceptually important step since it goes the other way around of what is normally performed. Instead of using the microsystem to cause quantum behavior of the mesoscopic system, we use the latter to enforce quantum

correlations on the former. Since a classical resonator is unable to generate entanglement among quantum systems coupled to it, this kind of study may work as a kind of classicality test in what concerns entanglement generation. We also study the qualitative behavior of tripartite entanglement as given by residual tripartite entanglement [17] and contrast its dynamical behavior to the ones of the bipartite entanglements [18]. Moreover, we investigate the regime where NEMS is initially in a thermal state, in such case one could expect that bipartite ion-ion entanglement would die for high temperature, but surprisingly there is not a temperature threshold above which they do not get entangled. Finally, we present results that show that in this system tripartite entanglement is more robust against energy relaxation than the bipartite entanglement.

The paper is organized as follows. Section II is dedicated to the presentation of the physical system studied here and Section III presents the theoretical tools necessary to understand the results presented in Section IV. Finally, Section IV D contains a summary of our results and conclusions. A appendix shows details about lengthy analytical expressions.

II. PHYSICAL SYSTEM AND MODEL

In this work, we consider a system consisting of two separate trapped ions capacitively coupled to a small doubly clamped nano beam (a NEMS resonator). The system is depicted in figure 1. The ions-NEMS coupling results from the application of an external bias voltage at an electrode on the NEMS (the bias gate is not shown in the picture).

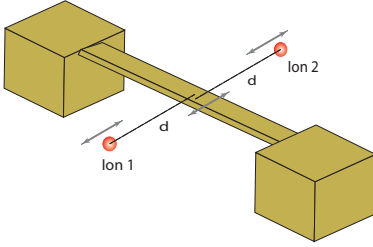


Figure 1: Sketch of the physical setup considered in this paper. Two trapped ions (traps not shown) interact with the flexural motion of a doubly clamped nano beam which is charged by a bias gate not shown in the picture.

The interaction energy between the motional degree of freedom of an ion with charge $+e$ and a NEMS with bias gate charge Q is given by the electrostatic coupling [7, 8]

$$V = \frac{keQ}{|d \pm (\hat{x} - \hat{X}_i)|} \quad (1)$$

where \hat{X}_i and \hat{x} refer to position operators of one ion and the NEMS, respectively, around their equilibrium positions. At

equilibrium, each ion is separated from the center of mass position of the NEMS by a distance d . Since we are dealing with positions around the equilibrium, the signs \pm are necessary to distinguish one ion from the other, for example, one can associate $+$ with ion 1 and $-$ with ion 2. Typically, one finds that $\langle \hat{X}_i \rangle, \langle \hat{x} \rangle \ll d$ so that the coupling energy can be expanded as

$$V \approx \frac{keV_0C_0}{d} \left\{ 1 \pm \frac{[\hat{X}_i - \hat{x}]}{d} + \frac{[\hat{X}_i - \hat{x}]^2}{d^2} \right\}, \quad (2)$$

with $Q = C_0V_0$, where C_0 and V_0 are the capacitance and gate voltage, respectively. According to 2, the coupling between the i_{th} ion and the NEMS is just $V = -\chi \hat{X}_i \hat{x}$, where $\chi = \frac{2keC_0V_0}{d^3}$. The linear and quadratic terms involving X_i or x alone have been absorbed into redefinitions of equilibrium positions and oscillators' frequencies.

Since the ions are not neutral particles, they will also couple by means of the same mechanism, i.e. and interaction term proportional to $\hat{X}_1 \hat{X}_2$ [19]. However, once the equilibrium distance between them is $2d$ (twice the distance to the NEMS) and we will be considering identical traps (identical oscillation frequencies), only resonant terms in $\hat{X}_1 \hat{X}_2$ will be relevant in this weak coupling regime (rotating wave approximation RWA) [19]. Considering $V = -\chi \hat{X}_i \hat{x}$ as the interaction between each ion and the NEMS and a weak RWA interaction between the ions, the Hamiltonian operator for this tripartite mechanical oscillating system reads ($\hbar = 1$)

$$\begin{aligned} \hat{H} = & \omega \hat{a}_0^\dagger \hat{a}_0 + \sum_{j=1,2} \nu_j \hat{a}_j^\dagger \hat{a}_j - \Omega (\hat{a}_2^\dagger \hat{a}_1 + \hat{a}_1^\dagger \hat{a}_2) \\ & - \frac{\sqrt{2}}{4} (\hat{a}_0 + \hat{a}_0^\dagger) \sum_{j=1,2} \kappa_j (\hat{a}_j + \hat{a}_j^\dagger), \end{aligned} \quad (3)$$

where $a_0(a_0^\dagger)$ and $a_i(a_i^\dagger)$ are the annihilation(creation) operators for the NEMS and ion i , respectively. Also, ν_i is the vibrational frequency of each ion (trap frequency), Ω is the RWA interaction strength between the ions, m_i and M the masses of the i_{th} ion and NEMS, respectively, and

$$\kappa_i = \sqrt{\frac{1}{m_i M \nu_i \omega}} \chi \quad (4)$$

is the coupling constant for interaction between the NEMS and the i_{th} ion.

It is worthwhile to notice that the vibrational motion of the ions are *directly* coupled by means of a beam-splitter-like interaction which is well known to not dynamically generate entanglement for direct products of thermal or coherent states [20]. In this situation, by putting such ions in interaction with the NEMS, initial creation of entanglement will be the result of the action of this third party acting as an entangling device. The key element for the initial creation of ion-ion entanglement is the presence of terms of the kind $\hat{a}_0 \hat{a}_i$ or $\hat{a}_0^\dagger \hat{a}_i^\dagger$ with strength κ_i providing an *indirect* squeezing interaction between the ions. This is an interesting problem because, being an indirect interaction, not all values of κ_i will lead to squeezing and entanglement in the ionic subsystem. Studying

the conditions for this to happen is one of the goals of this article. As a final remark, it is important to emphasize that Hamiltonian (3) is quadratic in the creation and annihilation operators and then preserves Gaussianity.

III. THEORETICAL TOOLS

In this section, we present the main tools used in this work. Basically, we are going to discuss some techniques to treat continuous variable systems in Gaussian states and also measures of entanglement suitable to study tripartite systems.

A. Gaussian Systems

In order to study the entanglement dynamics that results from the quadratic Hamiltonian (3), we represent it in terms of coefficients for the momenta and coordinate operators grouped together in the vector $\hat{R} = (\hat{x}_0, \hat{p}_0, \hat{x}_1, \hat{p}_1, \hat{x}_2, \hat{p}_2)^\top$ which contains momentum and position operators ordered in a convenient manner. The result is

$$\hat{H} = \frac{1}{2} \hat{R}^\top \mathcal{H} \hat{R} + K = \frac{1}{2} \sum_{i,j=0}^2 (\hat{x}_i \mathbf{U}_{ij} \hat{x}_j + \hat{p}_i \mathbf{T}_{ij} \hat{p}_j) + K, \quad (5)$$

with

$$\mathbf{U} = \begin{pmatrix} \omega & -\kappa_1/\sqrt{2} & \kappa_2/\sqrt{2} \\ -\kappa_1/\sqrt{2} & \nu_1 & -\Omega \\ \kappa_2/\sqrt{2} & -\Omega & \nu_2 \end{pmatrix}, \quad (6)$$

$$\mathbf{T} = \begin{pmatrix} \omega & 0 & 0 \\ 0 & \nu_1 & -\Omega \\ 0 & -\Omega & \nu_2 \end{pmatrix}, \quad (7)$$

and, finally, $K \equiv \frac{1}{2}(\omega + \nu_1 + \nu_2)$ is a constant. The operators composing the vector \hat{R} fulfill the usual canonical commutation relation: $[\hat{R}_j, \hat{R}_k] = iJ_{jk}$, where the symplectic (6×6) diagonal-block matrix J is given by:

$$J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}. \quad (8)$$

In order to explore the entanglement content of the present model, we present now (and in next subsection) the necessary mathematical formalism of Gaussian CV entanglement theory.

Choosing initial Gaussian states for the three oscillators, Hamiltonian (5) will preserve Gaussianity and we can extract all information about the system from the knowledge of the accompanying covariance matrix (CM) γ whose elements are given by

$$\gamma_{jk} = \frac{1}{2} \langle \hat{R}_j \hat{R}_k + \hat{R}_k \hat{R}_j \rangle - \langle \hat{R}_j \rangle \langle \hat{R}_k \rangle, \quad (9)$$

where $\langle \cdot \rangle$ denotes an expectation value. For a bipartite system composed of subsystems A and B [30], an entanglement measure called logarithmic negativity can be written as a function of the covariance matrix as [21]

$$N_{A|B} = -\frac{1}{2} \sum_j \ln \left[\min(1, \mu_j^{\top_B}) \right], \quad (10)$$

where $\mu_j^{\top_B}$ are the symplectic eigenvalues [31] of the covariance matrix γ^{\top_B} , evaluated after partial transposition of system B . This partial transposition is achieved by a local time inversion in the oscillators pertaining to subsystem B : $p \rightarrow -p$. For example, if the system AB is composed by *two* oscillators, $\gamma^{\top_B} = \mathcal{P} \gamma \mathcal{P}$, with

$$\mathcal{P} = \text{Diag}(1, 1, 1, -1). \quad (11)$$

In order to study the time evolution of entanglement in this system, we make explicit use of the fact that (3) preserves the Gaussian character of the global state, so that we just have to calculate $\gamma(t)$. For a system Hamiltonian of the form (5), the CM evolves as [22, 23]

$$\gamma(t) = E_t \gamma_0 E_t^\top, \quad (12)$$

where

$$E_t = \exp[J\mathcal{H}t] \quad (13)$$

is a symplectic matrix obeying $E_t^\top J E_t = J$. As a result, all closed system dynamics is dictated by the eigenvalues of $J\mathcal{H}$.

B. Tripartite Entanglement

Our main goal in this work is to study the dynamics of entanglement in the present system, especially because it is generated by the coupling of a nanoscale system to two atomic ions, forming an interesting platform for quantum technology experiments [7, 8]. Bipartite entanglement can be quantitatively addressed in a Gaussian system by using the tools developed in the last section. We now want to study the build up of genuine tripartite entanglement in this hybrid system. If one wants to address this issue quantitatively, the way is to start from the well known monogamy relation for pure states [17] and define a legitimate tripartite measurement called residual tripartite entanglement by using concurrence or logarithmic negativity. In the case of mixed states, this quantification has to be done by means of a convex-roof extension [24].

For the purposes of our investigation, a semiquantitative approach is sufficient since we are not aiming at a particular quantum information protocol where a given value of an entanglement measure has a precise meaning. Instead, we are interested in knowing whether or not there is legitimate tripartite entanglement in our system when some bipartite entanglement has vanished and also to investigate the robustness of bi- or tripartite entanglement against dissipation in the NEMS. We can then follow the same procedure taken in [25] which is based on multimode inseparability classification proposed by

Giedke et al. [26]. In our case this would correspond to check whether or not the negativity (10) vanishes for all bipartitions, *i.e.*, by assigning $\{0\}$ to the NEMS and $\{1, 2\}$ to the ions, we want to check when $N_{i|jk} \neq 0$ for all $i, j, k = 0, 1, 2$ with $i \neq j \neq k$ — according to [26], the system is genuinely tripartite entangled only if this is the case. Consequently, one can, for instance, study the resilience of tripartite entanglement against dissipation by investigating the quantity

$$\tau_{012} = \min[N_{0|12}, N_{1|02}, N_{2|01}]. \quad (14)$$

From the multimode inseparability classification [26] it is then clear that genuine tripartite entanglement ceases to exist as soon as $\tau_{012} = 0$.

C. Interaction with Environment

The most general linear Markovian evolution of a density operator $\hat{\rho}$ preserving its positivity and the Gaussianity is described by the master equation (written in the Lindblad form):

$$\frac{\partial \hat{\rho}}{\partial t} = \frac{1}{i} [\hat{H}, \hat{\rho}] - \frac{1}{2} \sum_k \left(\hat{L}_k^\dagger \hat{L}_k \hat{\rho} + \hat{\rho} \hat{L}_k^\dagger \hat{L}_k - 2 \hat{L}_k \hat{\rho} \hat{L}_k^\dagger \right), \quad (15)$$

with a *quadratic* Hamiltonian such as (5) and linear Lindblad superoperators \hat{L}_k

$$\hat{L}_k = \lambda_k \cdot J \hat{R}, \quad (16)$$

where J is given by (8) and λ_k is a complex vector. By defining the superoperators $D = \text{Re } \Upsilon$, and $\Gamma = J(\mathcal{H} - \text{Im } \Upsilon)$ where $\Upsilon = \sum_{k=1}^M \lambda_k \lambda_k^\dagger$, one can show that the covariance matrix (9) obeys now the following equation of motion [27]:

$$\frac{d\gamma}{dt} = \Gamma \gamma + \gamma \Gamma^\top + D, \quad (17)$$

whose solution satisfying the initial condition $\gamma(t=0) = \gamma_0$ is given by [27, 28]

$$\gamma(t) = e^{\Gamma t} \gamma_0 e^{\Gamma^\top t} + \int_0^t dt' e^{\Gamma(t-t')} D e^{\Gamma^\top(t-t')}. \quad (18)$$

When thinking in terms of Wigner functions in phase space, all environment influence is represented by Υ , with its real part (contained in Γ) responsible for dissipation (the $J\mathcal{H}$ term alone is the symplectic evolution) while $\text{Im } \Upsilon$ is responsible for diffusion.

In our system, the nanoscale motion of the NEMS is certainly more susceptible to loss and decoherence than the motion of the trapped ions. This happens because the former is subjected to coupling to phonons in the substrate where the nanobeam is fixed, while the ions are kept trapped by means of electromagnetic potentials in low pressure environments (almost no residual gases) where collisions are minimized. Then, we will consider

$$\hat{L}_1 = \sqrt{\zeta(\bar{N}+1)} \hat{a} \quad \text{and} \quad \hat{L}_2 = \sqrt{\zeta \bar{N}} \hat{a}^\dagger, \quad (19)$$

as well as

$$\lambda_1 = \sqrt{\frac{\zeta}{2}(\bar{N}+1)} (i, -1)^\top \quad \text{and} \quad \lambda_2 = -\sqrt{\frac{\zeta}{2}\bar{N}} (i, 1)^\top \quad (20)$$

where \bar{N} is the mean photon number of the reservoir which is related to the temperature of the substrate, and ζ is the damping constant. Even in our case where the Hamiltonian and Liouvillian are quadratic on quadrature operators, the solution of the problem is quite involved due to matrix exponentiation and integration in (18). Next subsection will be devoted to the discussion of some particular regimes where the solution is feasible either analytically or numerically.

IV. RESULTS

In what follows we present our results concerning entanglement dynamics in the hybrid ion-NEMS-ion system presented in last section. We will be considering physically motivated cases consisting of closed and open system dynamics with zero or finite temperature T . In particular, we will be considering identical traps and ion species leading to equal frequencies for the ions $\nu_1 = \nu_2 = \nu$ and also equal ion-NEMS separation implying in $\kappa_1 = \kappa_2 = \kappa$. In this symmetric scenario, the set of eigenvalues (or the spectrum) of $J\mathcal{H}$ is $\text{Spec}_{\mathbb{C}}(J\mathcal{H}) = \{\eta_{\pm}, \pi_{\pm}, \rho_{\pm}\}$, with

$$\begin{aligned} \eta_{\pm} &= \pm i\omega_{\pm} \\ \pi_{\pm} &= \frac{\pm i}{\sqrt{2}} \sqrt{\omega^2 + \omega_{\pm}^2 + \sqrt{[\omega^2 - \omega_{\pm}^2]^2 + 8\kappa^2 \omega \omega_{\pm}}} \\ \rho_{\pm} &= \frac{\pm i}{\sqrt{2}} \sqrt{\omega^2 + \omega_{\pm}^2 - \sqrt{[\omega^2 - \omega_{\pm}^2]^2 + 8\kappa^2 \omega \omega_{\pm}}} \end{aligned} \quad (21)$$

where we defined $\omega_{\pm} = \nu \pm \Omega$.

It is the κ -dependence of the quartet $\{\pi_{\pm}, \rho_{\pm}\}$ which will determine the type of dynamics. Complex conjugate pairs of eigenvalues in $\text{Spec}_{\mathbb{C}}(J\mathcal{H})$ indicate rotations whereas a real pair of eigenvalues with opposite signs indicates hyperbolic movement in the phase space, the same as squeezing. From (21) it is easy to see that the first two eigenvalues η_{\pm} are always purely imaginary. Besides, if $\kappa \leq \sqrt{\omega(\nu - \Omega)}$, $\text{Spec}_{\mathbb{C}}(J\mathcal{H})$ will consist only of pure imaginary eigenvalues indicating rotation in phase space. On the other hand, if $\kappa > \sqrt{\omega(\nu - \Omega)}$, we obtain a “mixed” dynamics of rotations and “squeezing” which is then capable of generating some entanglement in the system. The same dynamical behavior can be obtained with more compact equations if we establish $\omega = \nu - \Omega$. Since $\nu \gg \Omega$, in general, this corresponds to the limit of approximate resonance of natural frequencies of the ion vibrational motion and NEMS. In this case the above spectrum simplifies to

$$\text{Spec}_{\mathbb{C}}(J\mathcal{H}) = \left\{ \pm i\omega_{\pm}, \pm i\sqrt{\omega(\omega + \kappa)}, \pm i\sqrt{\omega(\omega - \kappa)} \right\}, \quad (22)$$

such that by taking $\kappa > \omega$, one will get a mixed dynamics (rotations+squeezing). We recall that generating squeezing is

a necessary condition to have entanglement from a separable initial coherent or thermal state following Gaussian evolutions. Therefore, we shall now fix $\kappa > \omega$. The matrix elements of E_t necessary to obtain (12) for this spectrum can be found in appendix A.

A. Closed system with $T = 0$

In this subsection, we will be considering an initial pure state consisting of the product of coherent states for the ions and the NEMS, this leads to the initial CM $\gamma_0 = \frac{1}{2}\mathbf{1}_6$, where $\mathbf{1}_6$ is the 6×6 identity matrix. We then evolve this initial covariance matrix according to (12). In order to facilitate the interpretation of the matrix elements, we arrange it in sectors representing each system and it is given by

$$\gamma(t) = \left(\begin{array}{c|c} \gamma_N & \mathbf{C} - \mathbf{C} \\ \hline \mathbf{C}^\top & \gamma_I \end{array} \right), \quad (23)$$

where

$$\gamma_I = \frac{1}{2} \begin{pmatrix} \mathbf{A}_I & \mathbf{C}_I \\ \mathbf{C}_I^\top & \mathbf{A}_I \end{pmatrix}. \quad (24)$$

The submatrices are shown in Appendix A. It is worthwhile to notice the independence of the system CM (23) on ω_+ which is the only parameter in (22) that contains information on the ion-ion coupling constant Ω . This independence may be understood as a consequence of the fact that ω_\pm appears solely in the imaginary eigenvalues (rotations) not affecting the second moments. We are then in a regime where *all entanglement is due to the coupling of the ions to the NEMS and not due the coupling between the ions*.

In figure 2, the NEMS-induced entanglement between the ions is presented as a function of time. This plot shows that the stronger the NEMS-ion coupling κ , the stronger the entanglement between the ions. This coupling constant is externally controlled by the experimentalist and ω is fixed by the NEMS fabrication. Basically, according to (4), κ can be enhanced by mere increasing of the gate voltage on the NEMS. It should be stressed that there is a lower bound on κ for this creation of entanglement to happen (κ must be stronger than ω). Consequently, figure 2 illustrates a central result in this paper. Two trapped ions initially uncorrelated and prepared in coherent states can become entangled by interacting with a nanoelectromechanical resonator (also prepared in a coherent state) as soon as the ion-NEMS coupling achieve a certain value, and this can be controlled by external voltage gate on the NEMS device.

B. Closed system with $T \neq 0$

A thermal state is a thermal equilibrium state of a quantum oscillator with frequency ω in contact with a thermal reservoir

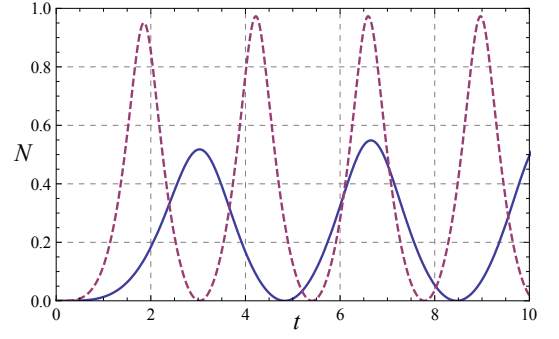


Figure 2: Logarithmic negativity for the ions when the system is initially prepared in a product of coherent states. Two coupling constants between the ions and the NEMS were considered: $\kappa = 1$ (solid) and $\kappa = 3$ (dashed). Also, $\omega = 0.5$ has been considered.

at temperature T (canonical ensemble). Its density matrix becomes diagonal in the Fock state basis $|m\rangle$:

$$\hat{\rho}_r = \frac{1}{\bar{n} + 1} \sum_{m=0}^{\infty} \left(\frac{\bar{n}}{\bar{n} + 1} \right)^m |m\rangle\langle m|, \quad (25)$$

where the mean phonon number is

$$\bar{n} = \left[\exp\left(-\frac{\omega}{k_B T}\right) - 1 \right]^{-1} \geq 0 \quad (26)$$

and k_B is the Boltzman constant. The CM of such state is written as $\gamma_r := (\bar{n} + \frac{1}{2})\mathbf{1}_2$. By defining $\alpha = 2\bar{n} + 1$ and still considering the ions in a product of coherent states, the initial CM of the total system is $\gamma_0^\alpha = \frac{1}{2}\text{Diag}(\alpha\mathbf{1}_2, \mathbf{1}_2, \mathbf{1}_2)$ and its evolution, following (12) and (22), will be

$$\gamma_\alpha(t) = \left(\begin{array}{c|c} \gamma_{\alpha N} & \mathbf{C}_\alpha - \mathbf{C}_\alpha \\ \hline \mathbf{C}_\alpha^\top & \gamma_{\alpha I} \end{array} \right) \quad (27)$$

with

$$\gamma_{\alpha I} = \frac{1}{2} \begin{pmatrix} \mathbf{A}_I^\alpha & \mathbf{C}_I^\alpha \\ \mathbf{C}_I^{\alpha\top} & \mathbf{A}_I^\alpha \end{pmatrix}. \quad (28)$$

It has the same structure of equation (23) but now \mathbf{C}_I^α and \mathbf{A}_I^α are lengthy and cumbersome expressions which we will not show explicitly. In the limit of $T \rightarrow 0$ ($\alpha \rightarrow 1$), we have checked analytically that $\gamma_\alpha(t) \rightarrow \gamma(t)$ which is given by (23).

Let us now investigate the entangling power of a thermal NEMS. Firstly, let us study the ion-ion dynamics for two different temperatures as shown in figure 3. We can see that by increasing the temperature (increasing α), two interesting trends appear. On one side, we can clearly see that the entanglement generated between the ions for a finite temperature is smaller than in the $T = 0$ case. In a certain way, this is expected since thermal fluctuations in the NEMS tend to turn it into a classical object. On the other hand, we can see that for a finite temperature there are collapses and revivals

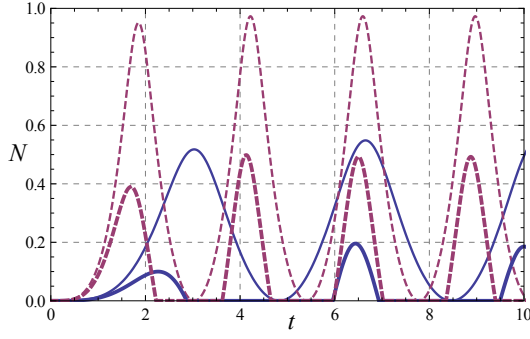


Figure 3: Logarithmic negativity for the ions when the NEMS is prepared in a thermal state characterized by α and the ions are initially prepared in a product of coherent states. Thin lines correspond to zero temperature ($\alpha = 1$) and thick lines correspond to a finite temperature ($\alpha = 5$). For the NEMS-ions coupling we used $\kappa = 1$ (solid) and $\kappa = 3$ (dashed). Also, $\omega = 0.5$ has been considered.

of entanglement in the dynamics. Also, we can see that an increase in temperature leads to a decrease in the maximal amount of entanglement generated between the ions. Numerical exploitation shows that there is not a temperature threshold above which no ion-ion bipartite entanglement is generated in the regime considered here. However, for higher temperatures the negativity remains non-zero (but very small) only for very short time intervals due to sudden death of entanglement and revivals. These trends can be seen in Fig.4. In general, the NEMS' entangling power degrades asymptotically as α grows, *i.e.*, as the NEMS becomes more classical.

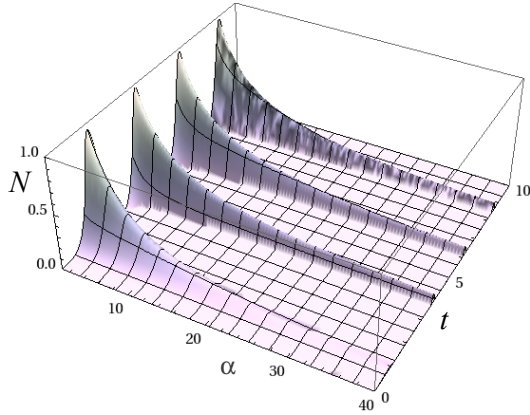


Figure 4: Logarithmic negativity for the ions as a function of time t and α which is proportional to the temperature of the NEMS (see main text). The ions were initially prepared in a product of coherent states. The other parameters are $\omega = 0.5$ and $\kappa = 3$.

C. Open System Dynamics and Tripartite Entanglement Resilience

We now study the case presented in Section III C where we briefly reviewed how dissipation in the NEMS can be taken into account in this Gaussian system. Our main goal is to in-

vestigate the resilience of bi- and tripartite entanglements in the system. To treat the genuine tripartite entanglement in our system, we use the methods explained in Section III B. In all the following plots, solid lines refer to the bipartite ion-ion entanglement, dashed lines refer to the bipartite NEMS-ion entanglement and dotted lines refer to the tripartite entanglement. Also, the initial system preparation is a product of coherent states for each system, just like in Section IV A. We would also like to emphasize that the bipartite entanglement between each ion and the NEMS are identical in the frequency regimes considered here, as previously explained.

In figure 5 we present the open system time evolution of bi- and tripartite entanglements for two different NEMS-ion couplings κ . The general behavior does not change with κ , except the maximal value of the entanglements which clearly increases with κ . From this plot, it is quite clear that the tripartite entanglement is much more resilient than the bipartite ones to losses in the NEMS. The latter goes to zero much later than the former as we can check numerically.

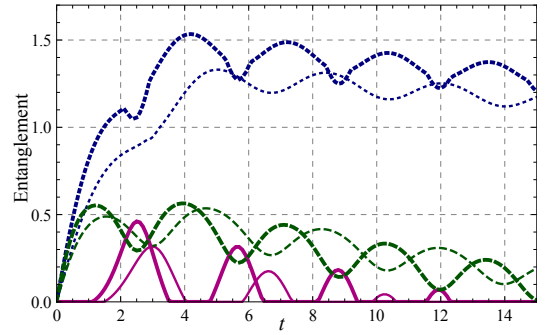


Figure 5: Open system dynamics for bi- and tripartite entanglements in the system consisting of one NEMS coupled to two trapped ions. Thin lines refer to $\kappa = 1$ and thick lines to $\kappa = 1.5$. The other system parameters are $\zeta = 0.01$ and $\omega = 0.5$. The thermal number of photons in the reservoir is $\bar{N} = 4.5$. Dotted lines corresponds to τ_{012} , dashed lines to N_{01} and continuous lines to N_{12} .

In order to further understand this apparent resilience of the tripartite entanglement when compared to the bipartite ones, we now present their time evolution for two different NEMS decay rates, keeping the reservoir temperature constant. This is presented in figure 6. As could be expected, the stronger the decay rates the stronger the suppression of entanglement. However, it is still remarkable that the tripartite entanglement takes a much longer time than the bipartite ones to go to zero, signaling again a pronounced robustness to coupling to the environment.

Now we analyze the thermal effect on the open system dynamics. The results for two different temperatures is shown in figure 7. Although thermal noise severely degrades entanglement, as expected, it is still noticeable that tripartite entanglement is more robust than the bipartite ones. An increase in temperature is more destructive to the bipartite entanglement in our system than to the tripartite one in the sense that the latter goes to zero much sooner than the former. We can see, for instance, that for $\bar{N} = 15$ no ion-ion bipartite entanglement appears, this shows that thermal fluctuations can erase this ef-

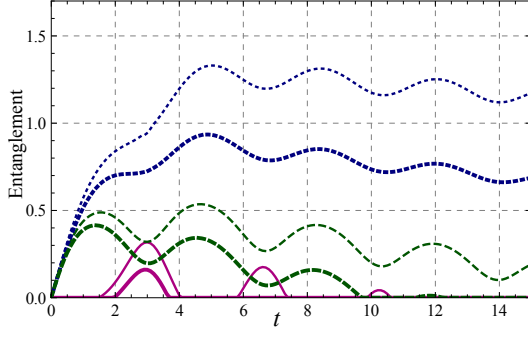


Figure 6: Open system dynamics for bi- and tripartite entanglements in the system for two different NEMS's decay constants. Thin lines refer to $\zeta = 0.01$ and thick lines to $\zeta = 0.02$. The other system parameters are $\kappa = 1$ and $\omega = 0.5$. The thermal number of photons in the reservoir is $\bar{N} = 4.5$. *Idem* fig.5.

fect of NEMS-induced ion-ion entanglement. Actually there is a threshold for the temperature of the bath, numerically we found that above $\bar{N} = 14.2$, no ion-ion is found (with precision 10^{-5}).

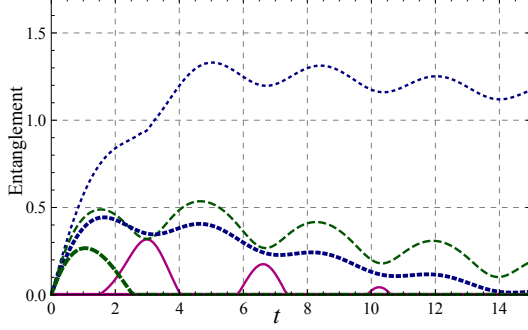


Figure 7: Open system dynamics for bi- and tripartite entanglements in the system for two different temperatures. Thin lines refer to $\bar{N} = 4.5$ and thick lines to $\bar{N} = 15$. The other system parameters are $k = 1$, $\omega = 0.5$ and $\zeta = 0.01$. *Idem* fig.5.

D. Conclusions

We have presented some results concerning entanglement generation in a tripartite system consisting of two atomic ions and a nanoscale mechanical resonator. We showed that the latter is capable of inducing entanglement in the former and we studied the features of such generation under ideal and non-ideal conditions. In the ideal case we show analytically the precise regime where the system becomes entangled or not. The amount of entanglement is controlled by the gate voltage applied on NEMS. Especially, we found that there is not a temperature threshold above which no bipartite ion-ion entanglement is created when we are dealing with the ideal case (no dissipation), there is only an asymptotically degradation of entanglement as the temperature of the NEMS' initial state grows. surprisingly a distinct effect occurs when the NEMS is coupled to an environment: correlations between them take

place and actually a temperature threshold appears. We have also found that the tripartite entanglement is much more resilient to losses in the NEMS than the bipartite ones as the latter goes to zero much later than the former. We believe our studies may be useful for deepening our understanding about the interaction of atomic and nanoscale systems, especially in what concerns the appearing and destruction of nonlocal quantum correlations under ideal and nonideal conditions.

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Appendix A: Some analytical matrix elements

In this appendix, we present the explicit analytical form of some matrix elements needed to obtain results in the main body of this manuscript.

The matrix elements of E_t (13) relative to the spectrum (22) are given by

$$\begin{aligned}
 E_{t11} &= E_{t44} = \frac{1}{2}(\cosh \tilde{\omega}t + \cos \bar{\omega}t), \\
 E_{t12} &= E_{t21} = E_{t45} = E_{t54} = \frac{\sqrt{2}}{4}(\cosh \tilde{\omega}t - \cos \bar{\omega}t), \\
 E_{t13} &= E_{t31} = E_{t46} = E_{t64} = -E_{t12}, \\
 E_{t22} &= E_{t55} = \frac{1}{4}(\cosh \tilde{\omega}t + \cos \bar{\omega}t + 2 \cos \omega_+t), \\
 E_{t23} &= E_{t32} = E_{t56} = E_{t65} \\
 &= \frac{1}{4}(2 \cos \omega_+t - \cos \bar{\omega}t - \cosh \tilde{\omega}t), \\
 E_{t33} &= E_{t66} = \frac{1}{4}(2 \cos \omega_+t + \cos \bar{\omega}t + \cosh \tilde{\omega}t), \\
 E_{t14} &= \frac{1}{2}\left(\frac{\omega}{\tilde{\omega}} \sin \bar{\omega}t + \frac{\omega}{\tilde{\omega}} \sinh \tilde{\omega}t\right), \\
 E_{t15} &= E_{t24} = \frac{\sqrt{2}}{4}\left(-\frac{\omega}{\tilde{\omega}} \sin \bar{\omega}t + \frac{\omega}{\tilde{\omega}} \sinh \tilde{\omega}t\right), \\
 E_{t16} &= E_{t34} = -E_{t15}, \\
 E_{t25} &= E_{t36} = \frac{1}{4}\left(\frac{\omega}{\tilde{\omega}} \sin \bar{\omega}t + \frac{\omega}{\tilde{\omega}} \sinh \tilde{\omega}t + 2 \sin \omega_+t\right), \\
 E_{t26} &= E_{t35} = -\frac{1}{4}\left(\frac{\omega}{\tilde{\omega}} \sin \bar{\omega}t + \frac{\omega}{\tilde{\omega}} \sinh \tilde{\omega}t - 2 \sin \omega_+t\right), \\
 E_{t41} &= \frac{1}{2}\left(-\frac{\omega}{\tilde{\omega}} \sin \bar{\omega}t + \frac{\omega}{\tilde{\omega}} \sinh \tilde{\omega}t\right), \\
 E_{t42} &= E_{t51} = \frac{\sqrt{2}}{4}\left(\frac{\omega}{\tilde{\omega}} \sin \bar{\omega}t + \frac{\omega}{\tilde{\omega}} \sinh \tilde{\omega}t\right), \\
 E_{t43} &= E_{t61} = -E_{t42}, \\
 E_{t52} &= E_{t63} = \frac{1}{4}\left(\frac{\omega}{\tilde{\omega}} \sinh \tilde{\omega}t - \frac{\omega}{\tilde{\omega}} \sin \bar{\omega}t - 2 \sin \omega_+t\right), \\
 E_{t53} &= E_{t62} = \frac{1}{4}\left(\frac{\omega}{\tilde{\omega}} \sin \bar{\omega}t - \frac{\omega}{\tilde{\omega}} \sinh \tilde{\omega}t - 2 \sin \omega_+t\right),
 \end{aligned}$$

where $\tilde{\omega} \equiv \sqrt{\omega(\kappa - \omega)}$, $\bar{\omega} \equiv \sqrt{\omega(\kappa + \omega)}$ and $\omega_+ := \Omega + \nu$.

The submatrices necessary to write the covariance matrix (23) are given by

$$\begin{aligned}\gamma_N &= \frac{1}{2} \begin{pmatrix} 1 + a(t) + b(t) & c(t) + d(t) \\ c(t) + d(t) & 1 + a'(t) + b'(t) \end{pmatrix}; \\ \mathbf{A}_I &= \begin{pmatrix} 1 + \frac{1}{2}[a(t) + b(t)] & \frac{1}{2}[c(t) + d(t)] \\ \frac{1}{2}[c(t) + d(t)] & 1 + \frac{1}{2}[a'(t) + b'(t)] \end{pmatrix}; \\ \mathbf{C}_I &= -\frac{1}{2} \begin{pmatrix} a(t) + b(t) & c(t) + d(t) \\ c(t) + d(t) & a'(t) + b'(t) \end{pmatrix}; \\ \mathbf{C} &= -\frac{\sqrt{2}}{4} \begin{pmatrix} a(t) - b(t) & c(t) - d(t) \\ c(t) - d(t) & a'(t) - b'(t) \end{pmatrix},\end{aligned}$$

where

$$a(t) = \frac{\omega^2 - \bar{\omega}^2}{2\bar{\omega}^2} \sin^2 \bar{\omega}t, \quad (\text{A1})$$

$$a'(t) = \frac{\bar{\omega}^2 - \omega^2}{2\omega^2} \sin^2 \bar{\omega}t, \quad (\text{A2})$$

$$b(t) = \frac{\omega^2 + \bar{\omega}^2}{2\bar{\omega}^2} \sinh^2 \bar{\omega}t, \quad (\text{A3})$$

$$b'(t) = \frac{\bar{\omega}^2 + \omega^2}{2\omega^2} \sinh^2 \bar{\omega}t, \quad (\text{A4})$$

$$c(t) = \frac{\omega^2 - \bar{\omega}^2}{4\omega\bar{\omega}} \sin 2\bar{\omega}t, \quad (\text{A5})$$

$$d(t) = \frac{\omega^2 + \bar{\omega}^2}{4\omega\bar{\omega}} \sinh 2\bar{\omega}t. \quad (\text{A6})$$

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- [1] D. Leibfried, R. Blatt, C. Monroe, and D. Wineland, *Rev. Mod. Phys.* **75**, 281 (2003).
 - [2] K. C. Schwab and M. L. Roukes, *Phys. Today* **58** (7), 36 (2005).
 - [3] G. Chen, D. A. Church, B. -G. Englert, C. Henkel, B. Rohwedder, M. O. Scully, and M. S. Zubairy, *Quantum Computing Devices: Principles, Designs, and Analysis* (Chapman & Hall/CRC, 2006).
 - [4] D. Rugar, R. Budakian, H. J. Mamin and B. W. Chui, *Nature (London)* **430**, 329 (2004).
 - [5] P. Rabl, S. J. Kolkowitz, F. H. L. Koppens, J. G. E. Harris, P. Zoller, and M. D. Lukin, *Nat. Phys.* **6**, 602 (2010).
 - [6] R. J. Schoelkopf, P. Wahlgren, A. A. Kozhevnikov, P. Delsing e D. E. Prober, *Science* **280**, 1238 (1998); B. Lassagne, D. Garcia-Sanchez, A. Aguasca e A. Bachtold, *Nano Lett.* **8**, 3735 (2008); D. Rugar, R. Budakian, H. J. Mamin e B. W. Chui, *Nature (London)* **430**, 329 (2004).
 - [7] L. Tian and P. Zoller, *Phys Rev Lett* **93**, 266403 (2004).
 - [8] W. K. Hensinger, D. W. Utami, H.-S. Goan, K. Schwab, C. Monroe, and G. J. Milburn, *Phys Rev A* **72**, 041405 (2005).
 - [9] M. Gao, Y. -X Liu, and X. -B. Wang, *Phys. Rev. A* **83**, 022309 (2011).
 - [10] J. D. Teufel, T. Donner, Dale Li, J. W. Harlow, M. S. Allman, K. Cicak, A. J. Sirois, J. D. Whittaker, K. W. Lehnert, and R. W. Simmonds, *Nature* **475**, 359 (2011).
 - [11] F. L. Semião, K. Furuya, G. J. Milburn, *Phys. Rev. A* **79**, 063811 (2009).
 - [12] M. Brune, E. Hagley, J. Dreyer, X. Maître, A. Maali, C. Wunderlich, J. M. Raimond, and S. Haroche, *Phys. Rev. Lett.* **77**, 4887 (1996); F. L. Semião and A. Vidiella-Barranco, *Phys. Rev. A* **71**, 065802 (2005).
 - [13] K. L. Ekinci, X. M. Huang, and M. L. Roukes, *Appl. Phys. Lett.* **84**, 4469 (2004).
 - [14] C.L. Degen, *Appl. Phys. Lett.* **92**, 243111 (2008).
 - [15] M. Poot and H. S.J. van der Zant, *Phys. Rep.* **511**, 273 (2012).
 - [16] J. Preskill, *Proc. R. Soc. Lond., Ser. A* **454**, 385 (1998); E. Knill, *Nature (London)* **463**, 441 (2010).
 - [17] V. Coffman, J. Kundu, and W. K. Wootters, *Phys. Rev. A* **61**, 052306 (2000); T. J. Osborne, and F. Verstraete, *Phys. Rev. Lett.* **96**, 220503 (2006); Y.-C. Ou and H. Fan, *Phys. Rev. A* **75**, 062308 (2007); D. S. Oliveira and R. V. Ramos, *Quantum Inf. Process.* **9**, 497 (2010).
 - [18] J. G. G. de Oliveira Jr., J. G. Peixoto de Faria, M. C. Nemes, *Phys Lett A* **375**, 4255 (2011).
 - [19] K. R. Brown, C. Ospelkaus, Y. Colombe, A. C. Wilson, D. Leibfried, and D. J. Wineland, *Nature* **471**, 196-199 (2011).
 - [20] W. Xiang-bin, *Phys Rev A* **66**, 024303 (2002).
 - [21] G. Vidal and R. F. Werner, *Phys. Rev. A*, **65**, 032314.
 - [22] M. B. Plenio, J. Hartley, and J. Eisert, *New J. Phys.* **6**, 36 (2004); M. B. Plenio and F. L. Semião, *New J. Phys.* **7**, 73 (2005).
 - [23] J. F. Leandro and F. L. Semião, *Phys Rev A* **79**, 052334 (2009).
 - [24] G. Adesso and F. Illuminati, *J. Phys. A* **40**, 7821 (2007).

- [25] J. Li, T. Fogarty, C. Cornick, J. Goold, T. Busch, and M. Paternostro, Phys. Rev A **84**, 022321 (2011).
- [26] G. Giedke, B. Kraus, M. Lewenstein, and J. I. Cirac, Phys. Rev. Lett. **87**, 167904 (2001); Phys. Rev. A **64**, 052303 (2001).
- [27] M. Wallquist, K. Hammerer, P. Zoller, C. Genes, M. Ludwig, F. Marquardt, P. Treutlein, J. Ye, and H. J. Kimble, Phys Rev A **81**, 023816 (2010).
- [28] C. Gardiner, *Stochastic Methods*, 4th ed. (Springer Verlag, Berlin, 2009).
- [29] M. de Gosson, *Symplectic Geometry and Quantum Mechanics* (Birkhäuser, Basel, series "Operator Theory: Advances and Applications", 2006).
- [30] In our case, partition A and B could be, for example, the NEMS and both ions, respectively. Also, we can trace out the NEMS and consider A to be ion 1 and B to be ion 2.
- [31] The symplectic eigenvalues of a positive definite matrix \mathbf{V} are given by the moduli of the (cartesian) eigenvalues of the matrix $iJ\mathbf{V}$ with J in (8). A demonstration could be found in [29].